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On VLSI interconnect optimization and linear ordering problem

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Abstract Some well-known VLSI interconnect optimizations problems for timing, power and cross-coupling noise immunity share a property that enables mapping them into a specialized Linear Ordering Problem (LOP). Unlike the general LOP problem which is NP-complete, this paper proves that the specialized one has a closed-form solution. Let $f(x,y):\mathbb{R}^2 \to \mathbb{R}$ be symmetric, non-negative, defined for $x \geq 0$ and $y \geq 0$, and let $f(x,y)$ be twice differentiable, satisfying $\partial^2 f(x,y)/\partial x \partial y < 0$. Let π be a permutation of $\{1,\dots,n\}$. The specialized LOP comprises n objects, each associated with a real value parameter r_i, $1 \leq i \leq n$, and a cost $f(r_i, r_j)$ associated to any two objects if $|\pi(i) - \pi(j)| = 1$, $1 \leq i, j \leq n$, and $f(r_i, r_j) = 0$ otherwise. We show that the permutation π which minimizes $\sum_{i=1}^{n-1} f(r_{\pi^{-1}(i)}, r_{\pi^{-1}(i+1)})$, called "*symmetric hill*", is determined upfront by the relations between the parameter values r_i.

Keywords VLSI interconnects optimization · Delay minimization · Power minimization · Linear ordering problem · Optimal permutation

1 Introduction and motivation

A VLSI interconnect model is shown in Fig. 1. There, logic gates called drivers drive signals that propagate along interconnecting wires. These signals stimulate other logic circuits, called receivers, connected at the opposite end of the wires. Cross-coupling parasitic capacitance which is the dominant cause for signal propagation

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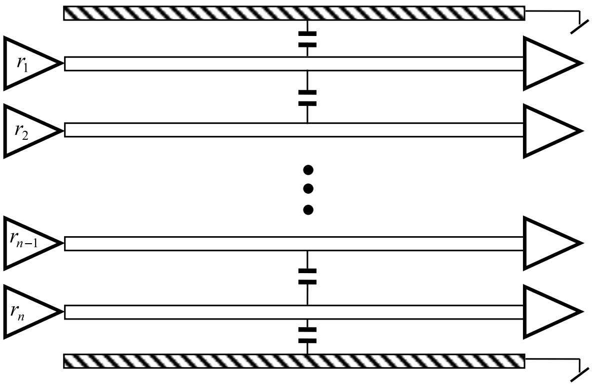

Fig. 1 Typical VLSI interconnects bus. Drivers are shown on the *left side* and receivers on the *right side*. The bus is shielded on its two sides. A parasitic cross-coupling capacitance is incurring between any adjacent signals

delay, power consumption and crosstalk noise interference, occurs only between adjacent wires. On both sides of the bus there are shielding wires connected to ground.

The authors of article (Macii et al. 2003) claimed by intuition that different orderings of the signal wires in Fig. 1 may yield different amounts of dynamic power consumption, and then proposed some monotonic order of the signals to reduce the power. Similarly, the authors of (Vittal et al. 1999) proposed an intuitive monotonic order aiming at reducing the noise interference. It was shown in (Moiseev et al. 2008a) and (Moiseev et al. 2008b) that both the total delay and power consumption are governed by an expression of the form:

$$F(r_1, \ldots, r_n) = \sqrt{r_1} + \sum_{i=1}^{i=n-1} \sqrt{r_i + r_{i+1}} + \sqrt{r_n}. \tag{1}$$

When delay minimization is concerned, the parameters $r_1, \ldots, r_n$ are derived from the resistances of the drivers in Fig. 1 and the parameters of manufacturing process technology (the lower resistance is, the more current is driven and the switching is faster). When power minimization is of interest, the parameters $r_1, \ldots, r_n$ represent the average amount of switching of a signal, called *activity factor*.

The problem of how to order the signals in the bus such that the expression in (1) yields minimum delay was addressed in (Moiseev et al. 2008a). It was shown that for the square root function a "*symmetric hill*" order is optimal. A similar order was obtained by (Moiseev et al. 2008b) for power minimization. This paper generalizes the above result by proving that the optimality of *symmetric hill* order exists for a broad type of functions, where square root is just a particular case. The technique of this proof can be used to derive different orders of objects (permutations) for different optimization goals. We'll discuss this further in the concluding section.

The permutation derived in this paper has been experimented on 65 nanometer process technology real VLSI design data. The experiments in (Moiseev et al. 2008a) showed potential of 10% wire delay reduction in chip's global interconnects, while those in (Moiseev et al. 2008b) showed potential of 17% dynamic power reduction. It is important to note that the optimal permutation by itself is not sufficient for delay and power deduction, but created better space sharing of adjacent wires. The optimization of the latter is a subject of so called *optimal wire spacing* (Cong et al. 2001).

Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be symmetric, non-negative, defined for $x \geq 0$ and $y \geq 0$, and let $f(x, y)$ be twice differentiable, satisfying:

$$\partial^2 f(x, y)/\partial x \partial y < 0. \tag{2}$$

Let $r_1, \ldots, r_n$ be n real non-negative numbers associated with n objects, and assume without loss of generality that $r_1 > r_2 > \cdots > r_n$. Let Π be the set of all permutations $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$. The ordered sequence $\langle i_1, \ldots, i_n \rangle$ is obtained by π, namely, $\pi(\langle 1, \ldots, n \rangle) = \langle i_1, \ldots, i_n \rangle$. We explore the problem of finding $\pi^* \in \Pi$ which minimizes the sum of costs defined for any two adjacent objects, given by:

$$F(\pi) = \sum_{j=1}^{n-1} f(r_{i_j}, r_{i_{j+1}}) \triangleq \sum_{j=1}^{n-1} f(r_{\pi^{-1}(j)}, r_{\pi^{-1}(j+1)}). \tag{3}$$

The problem in (3) is a type of Linear Ordering Problem (LOP) which is well known and used in economy and other applications. It has been studied extensively in the literature (Reinelt 1985; Laguna et al. 1999; Mitchell and Borchers 2000; Campos et al. 2001; Garcia et al. 2006). Given a $n \times n$ matrix of weights $C = (c_{ij})$, LOP aims at finding a permutation π which maximizes the expression $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{\pi(i)\pi(j)}$. LOP is NP-complete. In our setting the cost in (3) is derived from a function of two variables satisfying (2). It is defined for any two adjacent objects of the permutation, namely $|\pi(i) - \pi(j)| = 1$, $1 \leq i, j \leq n$, and is zero otherwise.

Equation (3) generalizes the objectives of the VLSI interconnects design optimization problems addressed in (Macii et al. 2003; Vittal et al. 1999; Moiseev et al. 2008a, 2008b). The terms $f(x, y)$ in (3) result from the cross-coupling capacitance, and they take the form $\sqrt{x + y}$ as mentioned before. The goal is therefore to determine the order of wires within the bus such that (3) is minimized. Notice that (1) is a special case of (3), as we could add an artificial object $r_0 = r_{n+1} = 0$ and then replace the first and last terms in the right hand side of (1) by $\sqrt{r_0 + r_1}$ and $\sqrt{r_n + r_{n+1}}$, respectively, thus resulting in a cyclical sum.

2 Minimizing a general objective function

We assume without loss of generality that the original set of objects is ordered such that $r_1 > r_2 > \cdots > r_n$. Let us modify (3) for the sake of proof convenience into a cyclical sum as follows:

$$F(\pi) = \sum_{j=1}^{n} f(r_{i_j}, r_{i_{j \bmod n+1}}) \triangleq \sum_{j=1}^{n} f(r_{\pi^{-1}(j)}, r_{\pi^{-1}(j \bmod n+1)}). \tag{4}$$

This doesn't change the original problem as we could add an artificial $(n + 1)$th zero object to (3). The sum in (4) is cyclical since the last index of permutation is interacting with the first one. Assume further that n is odd, since if it was even we could add a zero object as done in the square root example (1) for the first and last terms. In the rest of the discussion we'll drop modulo notation but keep in mind that summation is cyclical. The derivation of optimal permutation is based on the following two lemmas.

Lemma 1 *Let a, b, c and d be nonnegative real numbers satisfying $a > b > c > d \geq 0$ and let f satisfy* (2). *Then*:

$$f(a,b) + f(c,d) < f(a,c) + f(b,d) < f(a,d) + f(b,c). \tag{5}$$

Proof Consider first the left hand inequality of (5) which holds iff $f(a,b) - f(a,c) < f(b,d) - f(c,d) = f(d,b) - f(d,c)$, where the equality stems from the symmetry of f. Define $g(x) = f(x,b) - f(x,c)$. Then, left hand of (5) holds iff $g(a) < g(d)$. Since $b > c$, we have

$$\frac{\partial g(x)}{\partial x} = \frac{\partial[f(x,b) - f(x,c)]}{\partial x} = \int_c^b \frac{\partial^2 f(x,y)}{\partial x \partial y} dy < 0, \tag{6}$$

implying that $g(x)$ is monotonic decreasing. Since $a > d$, it follows that $g(a) < g(d)$, as desired. The right hand inequality of (5) follows analogously by noticing that it holds iff $f(a,c) - f(a,d) < f(b,c) - f(b,d)$, and defining $g(x) = f(x,c) - f(x,d)$. □

Lemma 2 *Let f satisfy* (2). *Any permutation $\pi^* \in \Pi$ which minimizes* (4) *must satisfy $|\pi^*(1) - \pi^*(2)| = |\pi^*(1) - \pi^*(3)| = 1$, namely, $\pi^*(2)$ and $\pi^*(3)$ must be adjacent to $\pi^*(1)$ on its two opposite sides in the sequence $\pi(\langle 1, \ldots, n \rangle) = \langle i_1, \ldots, i_n \rangle$, thus implying that the terms $f(r_1, r_2)$ and $f(r_1, r_3)$ must exist in the minimal sum*:

$$F(\pi^*) = \min_{\pi \in \Pi} \left\{ \sum_{j=1}^{n} f(r_{i_j}, r_{i_{j+1}}) \right\} \triangleq \min_{\pi \in \Pi} \left\{ \sum_{j=1}^{n} f(r_{\pi^{-1}(j)}, r_{\pi^{-1}(j+1)}) \right\} \tag{7}$$

Proof Let π^* satisfy (7), resulting in the ordered sequence $\langle i_1^*, \ldots, i_n^* \rangle$, and assume in contrary that $\pi^*(2)$ is not adjacent to $\pi^*(1)$ in $\langle i_1^*, \ldots, i_n^* \rangle$, say $\pi^*(1) = i_p^*, \pi^*(2) = i_{p+k}^*, k > 1$. Consider the non-empty subsequence $\langle i_p^*, i_{p+1}^*, \ldots, i_{p+k}^*, i_{p+k+1}^* \rangle$. Let π^{**} be a permutation, $\pi^{**}(\langle 1, \ldots, n \rangle) = \langle i_1^{**}, \ldots, i_n^{**} \rangle$, obtained from π^* by reverting (flipping) the order of the internal elements in that subsequence into $\langle i_p^*, i_{p+k}^*, \ldots, i_{p+1}^*, i_{p+k+1}^* \rangle$. The element adjacencies in π^{**} agree with π^* except those involving i_{p+1}^* and i_{p+k}^*. Let us subtract $F(\pi^{**})$ from $F(\pi^*)$. All identical terms are then canceled out except those involving i_{p+1}^* and i_{p+k}^*. It follows from the optimality of π^* in (7) that:

$$\begin{aligned} 0 > F(\pi^*) - F(\pi^{**}) &= \sum_{j=1}^{n} f(r_{i_j^*}, r_{i_{j+1}^*}) - \sum_{j=1}^{n} f(r_{i_j^{**}}, r_{i_{j+1}^{**}}) \\ &= [f(r_{i_p^*}, r_{i_{p+1}^*}) + f(r_{i_{p+k}^*}, r_{i_{p+k+1}^*})] - [f(r_{i_p^*}, r_{i_{p+k}^*}) + f(r_{i_{p+1}^*}, r_{i_{p+k+1}^*})], \end{aligned}$$

which results in the following inequality:

$$f(r_{i^*_p}, r_{i^*_{p+k}}) + f(r_{i^*_{p+1}}, r_{i^*_{p+k+1}}) > f(r_{i^*_p}, r_{i^*_{p+1}}) + f(r_{i^*_{p+k}}, r_{i^*_{p+k+1}}). \tag{8}$$

The initial setting $r_1 > r_2 > \cdots > r_n$, $\pi^*(1) = i^*_p$ and $\pi^*(2) = i^*_{p+k}$ implies $r_{i^*_p} > r_{i^*_{p+1}}$, $r_{i^*_p} > r_{i^*_{p+k+1}}$, $r_{i^*_{p+k}} > r_{i^*_{p+1}}$ and $r_{i^*_{p+k}} > r_{i^*_{p+k+1}}$. Setting $a = r_{i^*_p}$, $b = r_{i^*_{p+k}}$, $c = \max\{r_{i^*_{p+1}}, r_{i^*_{p+k+1}}\}$ and $d = \min\{r_{i^*_{p+1}}, r_{i^*_{p+k+1}}\}$, there exists $a > b > c > d \geq 0$ and the conditions of Lemma 1 are satisfied as follows. If it happened that $c = r_{i^*_{p+1}}$ and $d = r_{i^*_{p+k+1}}$ then substitution in left hand of inequality (5) yields $f(r_{i^*_p}, r_{i^*_{p+k}}) + f(r_{i^*_{p+1}}, r_{i^*_{p+k+1}}) < f(r_{i^*_p}, r_{i^*_{p+1}}) + f(r_{i^*_{p+k}}, r_{i^*_{p+k+1}})$, which contradicts (8). If it happened however that $c = r_{i^*_{p+k+1}}$ and $d = r_{i^*_{p+1}}$, substitution in the two ends of inequality (5) yields again the same contradiction. In conclusion there must exist $|\pi^*(1) - \pi^*(2)| = 1$.

It can be similarly shown that $|\pi^*(1) - \pi^*(3)| = 1$, where r_3 and r_2 reside on the opposite sides of r_1. Notice that the cyclical order of the objects yields two subsequences in π^* between $\pi^*(1)$ and $\pi^*(2)$, and two between $\pi^*(1)$ and $\pi^*(3)$, thus was enabling the selection of two disjoint subsequences for inversion such that they do not interfere with each other. □

Theorem 1 *Let f satisfy (2) and F be defined in (3). Let n be odd and $r_1, \ldots, r_n$ be n non-negative real numbers satisfying $r_1 > r_2 > \cdots > r_n$. Then the permutation $\pi^*(\langle 1, 2, \ldots, n\rangle) = \langle n, n-2, \ldots, 3, 1, 2, 4, \ldots, n-3, n-1\rangle$, defined by*:

$$\pi^*(i) = \begin{cases} \frac{n+1+i}{2} & i \text{ is even}, \\ \frac{n+2-i}{2} & i \text{ is odd}, \end{cases} \tag{9}$$

satisfies:

$$F(\pi^*) = \min_{\pi \in \Pi} \sum_{j=1}^{n} f(r_{i_j}, r_{i_{j+1}}), \tag{10}$$

where $\pi(\langle 1, \ldots, n\rangle) = \langle i_1, \ldots, i_n\rangle$.

Proof The proof follows by induction. Let $\pi^*(\langle 1, \ldots, n\rangle) = \langle i^*_1, \ldots, i^*_n\rangle$. We use the convention that in the cyclical order of the permutation (9) even indices are added one by one counterclockwise on one side of the maximal object r_1, while odd indices are added one by one clockwise on its opposite side, as shown in Fig. 2. Since the sequence is cyclical, we set arbitrarily the value $\pi^*(1) = (n+1)/2$, which is consistent with (9). Lemma 2 proved that $\pi^*(2)$ and $\pi^*(3)$ must be adjacent to $\pi^*(1)$ on its two opposite sides. This also satisfies (9) since $\pi^*(2) = (n+3)/2$ and $\pi^*(3) = (n-1)/2$, obtaining the counterclockwise successiveness of $\pi^*(3)$, $\pi^*(1)$ and $\pi^*(2)$.

Let $p < n$ be the smallest index of the original sequence for which $\pi^*(p)$ does not satisfy (9). Assume without loss of generality that p is even. The satisfaction of the induction hypothesis up to $p-1$ implies that π^* results the following sequence:

$$\pi^*(\langle 1, \ldots, n\rangle) = \langle \ldots, i^*_s, p-1, \ldots, 3, 1, 2, \ldots, p-2, i^*_t, \ldots\rangle, \tag{11}$$

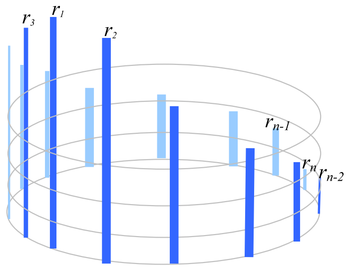

Fig. 2 Symmetric Hill Order. The largest elements reside at the hill, while the smallest ones reside at the valley

where $i_t^* \neq p$. Since the sequence is cyclic p is positioned somewhere in the following counterclockwise complementary subsequence $\langle p-2, i_t^*, \ldots, p, i_q^*, \ldots, i_s^*, p-1\rangle$. Notice that here may be two cases, $p \neq i_s^*$ or $p = i_s^*$, a case where $\langle p, i_q^* \ldots, i_s^*\rangle$ is just the single index p.

Let us transform π^* into π^{**} by reverting (flipping) $\langle i_t^*, \ldots, p\rangle$ into $\langle p, \ldots, i_t^*\rangle$, thus resulting in the subsequence $\langle p-2, p, \ldots, i_t^*, i_q^*, \ldots, i_s^*, p-1\rangle$. As in the proof of Lemma 2 we'll derive a contradiction by evaluating $F(\pi^*) - F(\pi^{**})$ on one hand, and applying direct substitution of the explicit permutations on the other hand.

Consider first the case $p \neq i_s^*$. The initial setting $r_1 > r_2 > \cdots > r_n$ implies $r_{p-2} > r_{i_t^*}, r_{p-2} > r_{i_q^*}, r_p > r_{i_t^*}$ and $r_p > r_{i_q^*}$. Setting $a = r_{p-2}, b = r_p, c = \max\{r_{i_t^*}, r_{i_q^*}\}$ and $d = \min\{r_{i_t^*}, r_{i_q^*}\}$ implies $a > b > c > d \geq 0$, which satisfies the conditions of Lemma 1. As in Lemma 2, either $f(a,b) + f(c,d) < f(a,c) + f(b,d)$ or $f(a,b) + f(c,d) < f(a,d) + f(b,c)$ of (5) holds. Therefore, here exist:

$$F(\pi^*) - F(\pi^{**}) = [f(r_{p-2}, r_{i_t^*}) + f(r_p, r_{i_q^*})] - [f(r_{p-2}, r_p) + f(r_{i_t^*}, r_{i_q^*})] > 0,$$

namely, $F(\pi^*) > F(\pi^{**})$, which contradicts the optimality of π^*.

Consider now the case $p = i_s^*$. Setting $a = r_{p-2}, b = r_{p-1}, c = r_p$ and $d = r_{i_t^*}$, it follows again that $a > b > c > d \geq 0$, which satisfies the conditions of Lemma 1 and $f(a,c) + f(b,d) < f(a,d) + f(b,c)$ of (5) holds. Therefore, here exist:

$$F(\pi^*) - F(\pi^{**}) = [f(r_{p-2}, r_{i_t^*}) + f(r_{p-1}, r_p)] - [f(r_{p-2}, r_p) + f(r_p, r_{i_t^*})] > 0,$$

concluding again that $F(\pi^*) > F(\pi^{**})$, which is a contradiction. □

Figure 2 illustrates the *symmetric hill* optimal permutation proved by Theorem 1 to minimize the sum of functions evaluated for cyclically ordered adjacent objects. The optimal order has one peak (maximum) and one valley (minimum) located oppositely to each other, while all elements are evenly distributed on both sides, which resembles a symmetric hill. This can also be seen by closing the ends of the sequence $\langle n, n-2, \ldots, 3, 1, 2, 4, \ldots, n-3, n-1\rangle$, turning it into cycle.

Symmetric hill permutation guarantees that the two shields of Fig. 1 are positioned in the valley since their corresponding parameters are practically zero. (Shields are not switching, hence have zero driver strength and zero activity factor.) In the initial setting $r_1 > r_2 > \cdots > r_n$ their value can be set arbitrary small, being identified with r_{n-1} and r_n. In terms of VLSI planar layout it means that they stay at the boundary of the bus.

3 Conclusions

The existence of a closed-form solution for a specialized LOP has been proven, where the cost associated with any two objects obeys a general two-variable function that was found useful for several optimization problems arising in VLSI interconnect design. The optimal solution is obtained by a unique permutation of the objects called *symmetric hill*. This order can be derived directly from the problem setting since it depends only on the relations between the parameter associated to the objects.

The technique used in this paper may be applicable to other objective functions, which may yield different permutations that can be defined upfront based on the relations between the values of objects' parameters. The authors believe that under the same setting of LOP, a maximization will yield a monotonic jigsaw permutation. Furthermore, the idea that the cost associated with two objects satisfies a type of function may be useful for other permutation problems such as quadratic assignment.

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